# Degeneracy of Holomorphic Maps Omitting Hypersurfaces (超曲面の特異点)



SAKAI, FUMIO

数理解析研究所講究録
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# DEGENERACY OF HOLOMORPHIC MAPS OMITTING HYPERSURFACES

Fumio Sakai

Let W be a projective algebraic manifold of dimension n and D a hypersurface on W. Let $f:\mathbb{C}^n \longrightarrow W-D$ be a holomorphic map. We say that f is degenerate if the Jacobian of f vanishes identically. In this note, we shall deal with the influence of the singularity of D on degeneracy theorems of f.

## 1. Notations

A hypersurface D on W is said to have simple normal crossings if each irreducible component of D is non-singular and D has normal crossings, i.e., D is locally given by $w_1 \cdots w_j=0$, where $(w_1,\ldots,w_n)$ are local coordinates of W.

Let L be a line bundle on W. Let $h^i(L)=\dim H^i(W,\mathcal{O}(L))$. The L-dimension $\kappa(L,W)$ of W is roughly the polynomial order of $h^0(mL)$ as a function of positive integers m. Note that $\kappa(L,W)$ takes one of the values $-\infty,0,1,\ldots,n$. Here we need the following fact: $\kappa(L,W)=n$ if and only if

$$\limsup_{m\to+\infty} m^{-n}h^0(mL)>0.$$

If $c_1(L)>0$, then $\kappa(L,W)=n$. For a divisor D, we denote by [D] the associated line bundle. We write $\kappa(D,W)=\kappa([D],W)$.

## 2. Degeneracy theorem

Picard's theorem states that any holomorphic map $f:\mathbb{C}\longrightarrow\mathbb{P}_1$

omitting three points is a constant map. We begin with the following generalization of this theorem.

<u>Theorem</u> 1 ([15], see also [2],[10]). Let W be a projective algebraic manifold of dimension n and D a hypersurface on W. Suppose that

(i) $\kappa(K_W+D,W)=n$, where $K_W$ is the <u>canonical</u> bundle of W,

(ii) D has simple normal crossings.

Then any holomorphic map $f:\mathbb{C}^n \longrightarrow W-D$ is degenerate.

<u>Remark</u>. In case $W=\mathbb{P}_n$ and D= a hypersurface of degree d, the hypothesis (i) is satisfied if and only if $d\geqq n+2$.

The following example shows that we cannot remove the hypothesis (ii).

<u>Example</u> 1. Let $W=\mathbb{P}_2$ and $D=\{w_0^{d-1}w_2-w_1^d=0\}$, where $[w_0:w_1:w_2]$ are homogeneous coordinates of $\mathbb{P}_2$. By the above remark, if $d\geqq 4$, the hypothesis (i) is satisfied. D has only one singularity at $[0:0:1]$. Define a holomorphic map $f:\mathbb{C}^2=(z_1,z_2) \longrightarrow \mathbb{P}_2$ by $f(z_1,z_2)=[1:z_1:z_1^d+e^{z_2}]$. Then f omits D and the Jacobian of f is

$$J_f=\begin{vmatrix} 1 & 0 \\ dz_1^{d-1} & e^{z_2} \end{vmatrix}=e^{z_2}\neq 0.$$

In what follows, we shall consider the question: what happens when D has worse singularities than simple normal crossings in Theorem 1?

## 3. <u>Resolution of sigularities</u>

Let W be a projective algebraic manifold of dimension n and

$D$ a hypersurface on $W$. If $D$ does not satisfy the hypothesis (ii) in Theorem 1, by desingularizing $D$, we can find $W^*$ and $D^*$ satisfying the following conditions:

(i) $\pi : W^* \longrightarrow W$ is a composite of monoidal transformations,

(ii) $\pi : W^* - D^* \longrightarrow W - D$ is biholomorphic,

(iii) $D^* =$ the support of $\pi^* D$,

(iv) $D$ has simple normal crossings.

From Theorem 1, it follows

Theorem 2. If $\kappa(K_{W^*} + D^*, W^*) = n$, then any holomorphic map $f : \mathbb{C}^n \longrightarrow W - D$ is degenerate.

Proof. It suffices to consider $f$ as a holomorphic map to $W^* - D^*$, q.e.d.

$$\begin{tikzcd} & W^* - D^* \arrow[d, "\pi"] \\ f : \mathbb{C}^n \arrow[ur] \arrow[r] & W - D \end{tikzcd}$$

To calculate $\kappa(K_{W^*} + D^*, W^*)$, we study the process of the desingularization, in which we have a sequence of monoidal transformations $\pi_i : W_{i+1} \longrightarrow W_i$ with non-singular centers $C_i$ such that

(i) $W_0 = W$, $W_\ell = W^*$,

(ii) $D_i =$ the support of $\pi^*_{i-1} D_{i-1}$,

(iii) $D_\ell = D^*$, which has simple normal crossings.

$$\begin{tikzcd} W^* = W_\ell \arrow[d] & \supset & D_\ell = D^* \arrow[d] \\ \vdots \arrow[d] & & \vdots \arrow[d] \\ W_1 \arrow[d, "\pi_0"'] & \supset & D_1 \arrow[d, "\pi_0"] \\ W = W_0 & \supset & D_0 = D \end{tikzcd}$$

Define

$\bar{D}_i =$ the strict transform of $D_{i-1}$ by $\pi_{i-1}$,

$E_i =$ the exceptional locus of $\pi_{i-1}$, i.e., $\pi^{-1}_{i-1}(C_{i-1})$,

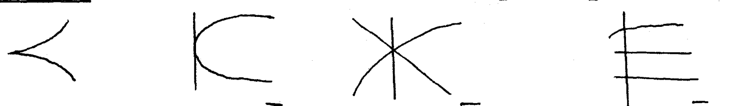
$\delta_i =$ the codimension of $C_i$ in $W_i$,

$\nu_i$=the multiplicity of the singular locus of $D_i$ at $C_i$.

Then we have

$$(1)\quad \begin{aligned} &D_i=\bar{D}_i+E_i, \quad \pi^*_{i-1}D_{i-1}=\bar{D}_i+\nu_{i-1}E_i, \quad K_{W_i}=\pi^*_{i-1}K_{W_{i-1}}+[(\delta_{i-1}-1)E_i], \\ &K_{W_i}+[D_i]=\pi^*_{i-1}(K_{W_{i-1}}+[D_{i-1}])+[(\delta_{i-1}-\nu_{i-1})E_i] . \end{aligned}$$

Example 2. We examine the cusp $D=\{y^2=x^3\}$.



$D$ $\qquad$ $E_1$ $\bar{D}_1$ $\qquad$ $E_2$ $\bar{D}_2$ $\qquad$ $E_3$ $\bar{D}_3$

$\nu_0=2$ $\qquad$ $\nu_1=2$ $\qquad$ $\nu_2=3$

Propsition 1. $\kappa(K_{W*}+D^*,W^*)\leqq\kappa(K_W+D,W)$.

This follws from properties (1), and further if D has simple normal crossings, the equality holds. Moreover we have the following

Proposition 2. Let $f:V'\longrightarrow V$ be a birational morphism, where $V',V$ are projective algebraic manifolds. Let D be a hyper-surface on V and put $D'$=the support of $f^*D$. Then

$$\kappa(K_{V'}+D',V')\leqq\kappa(K_V+D,V).$$

Remark. $\kappa(K_{W*}+D^*,W^*)$ is independent of the desingulariztion $W^*,D^*$. In fact let $W_1,D_1$ be another desingularization of D. There exists a desingularization $W^{**},D^{**}$, which is obtained by a sequence of monoidal transformations of $W^*$, with centers over $D^*$, such that there is a birational morphism $\phi:W^{**}\longrightarrow W_1$. The assertion follows from the above propositions.

Definition. We say that D has quasi-negligible singularities if $\nu_i \leqq \delta_i$ holds for $i=0,\ldots,\ell-1$.

Proposition 3 ([15]). If D has quasi-negligible singularities, then $\kappa(K_{W^*}+D^*,W^*)=\kappa(K_W+D,W)$.

Proof. By (1), we have

$$\kappa(K_{W_i}+D_i,W_i) \geqq \kappa(\pi_{i-1}^*(K_{W_{i-1}}+D_{i-1}),W_i)=\kappa(K_{W_{i-1}}+D_{i-1},W_{i-1}).$$

Hence we have $\kappa(K_{W^*}+D^*,W^*) \geqq \kappa(K_W+D,W)$, which proves Proposition 3.

Thus the hypothesis (ii) in Theorem 1 can be weakened as:

(ii)* D has quasi-negligible singularities.

Examples of quasi-negligible singularities

(i) normal crossing is quasi-negligible,

(ii) a curve has quasi-negligible singularities if and only if its singularities are ordinary double points,

(iii) the isolated singularity $w_1^d+\cdots+w_n^d=0$ is quasi-negligible if $d \leqq n$ (this type appeared in Carlson [1]),

(iv) on surfaces the singularity defined by $w_1^2+w_2^2+w_3^{k+1}=0$ (type $A_k$) is quasi-negligible.

Proposition 4 ([15]). If the Kodaira dimension $\kappa(W)=\kappa(K_W,W) \geqq 0$, then we have $\kappa(K_{W^*}+D^*,W^*)=\kappa(K_W+D,W)$.

Therefore in case $\kappa(W) \geqq 0$, the hypothesis (ii) can be removed. This leads us to study the case $\kappa(W)<0$. Note that $\kappa(W)<0$ if and only if $h^0(mK_W)=0$, for every positive integer m.

In case $n=2$, a surface $S$ with $\kappa(S)<0$ is birationally equivalent to $\mathbb{P}_1 \times C$, where $C$ is a curve (ruled surface). We have

Proposition 5. Let $S$ be an algebraic surface and $D$ a curve on $S$. If $(K_S+D)^2-(\nu_0-2)^2-\cdots-(\nu_{\ell-1}-2)^2>0$, then $\kappa(K_S+D,S)=2$ implies $\kappa(K_{S*}+D*,S*)=2$.

Proof. By (1), putting $\delta_i=2$, we get

$$(K_{S_i}+D_i)^2=(\pi^*_{i-1}(K_{S_{i-1}}+D_{i-1})+(2-\nu_{i-1})E_i)^2$$
$$=(K_{S_{i-1}}+D_{i-1})^2-(\nu_{i-1}-2)^2.$$

Hence we obtain

$$(K_{S*}+D*)^2=(K_S+D)^2-(\nu_0-2)^2-\cdots-(\nu_{\ell-1}-2)^2>0.$$

Let $\Gamma=K_S+[D]$, $\Gamma*=K_{S*}+[D*]$. Using this notations we have $\Gamma*^2>0$. We infer from the Riemann-Roch theorem that

$$h^0(m\Gamma*)+h^2(m\Gamma*)\geqq\tfrac{1}{2}m\Gamma*(m\Gamma*-K_{S*})+p_a(S*).$$

Note that $h^2(m\Gamma*)=h^0(K_{S*}-m\Gamma*)\leqq h^0(-(m-1)\Gamma*)$. Thus the above inequality shows that either $h^0(m\Gamma*)>0$, or $h^0(-(m-1)\Gamma*)>0$, for large $m$. By (1), $\Gamma*=\pi*(\Gamma)-[\mathcal{E}]$, $\mathcal{E}$ is an exceptional divisor of $\pi$. If $h^0(-(m-1)\Gamma*)>0$, then $h^0(-(m-1)(\Gamma*+[\mathcal{E}]))=h^0(-(m-1)\Gamma)>0$, which is a contradiction. Hence $h^0(m\Gamma*)\geqq\frac{1}{2}(\Gamma*)^2m^2+\cdots$, which proves $\kappa(\Gamma*,S*)=2$, q.e.d.

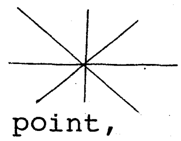
Example 3. Let $S=\mathbb{P}_2$ and $D=$ a hypersurface of degree $d$. The hypothesis in the above proposition is

$$(d-3)^2-(\nu_0-2)^2-\cdots-(\nu_{\ell-1}-2)^2>0.$$

Let $\kappa*=\kappa(\Gamma*,S*)$. We give some examples:

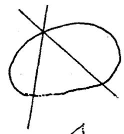
(i) four lines meeting at one point,

$$\Gamma^{*2} = -1, \quad \kappa^{*} = -\infty,$$



(ii) a conic and two lines meeting at one point,

$$\Gamma^{*2} = 0, \quad \kappa^{*} = 1,$$

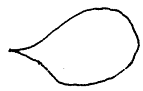


(iii) a quintic with two cusps,

$$\Gamma^{*2} = 2, \quad \kappa^{*} = 2,$$

(iv) the curve in Example 1,

$$\Gamma^{*2} < 0, \quad \kappa^{*} = -\infty .$$



Remark. Let $V'$, $V$ be projective algebraic manifold of dimension n and let $f: V' \longrightarrow V$ be a finite morphism. For a hypersurface D on V, let $D'$ =the support of $f^*D$. Then it is easily seen that $\kappa(K_{V'}+D', V') \geqq \kappa(K_V+D, V)$. Let $V'^*, D'^*$ and $V^*, D^*$ be desingularizations of $V', D'$ and $V, D$, respectively. Is it true that

$$\kappa(K_{V'^*}+D'^*, V'^*) \geqq \kappa(K_{V^*}+D^*, V^*) \quad ?$$

## 4. Concluding Remarks

Let M be a complex manifold of dimension n. We define the following properties of M.

$(ED)_k$ Every holomorphic map $f: \mathbb{C}^k \times \mathbb{D}^{n-k} \longrightarrow M$ is degenerate, where $\mathbb{D}$ is the unit disk $\{z \mid |z| < 1\}$,

$(HD)_k$ Every holomorphic map $f: \mathbb{C}^k \longrightarrow M$ is degenerate in the sense that the rank of the Jacobian matrix of f is not maximal anywhere.

$(AD)_k$ Every holomorphic map $f:\mathbb{C}^k \longrightarrow M$ is algebraically degenerate, i.e., the image $f(\mathbb{C}^k)$ is contained in a proper subvariety of M.

Obviously we have the following relations:

$$\begin{array}{ccc} (HD)_k & \longrightarrow & (ED)_k \\ \updownarrow & & \updownarrow \\ (HD)_{k+1} & \longrightarrow & (ED)_{k+1} \end{array} \qquad (HD)_n=(ED)_n$$

The proof of Theorem 1 ([15]) implies the following stronger form of Theorem 1. (The conclusion of Theorem 1 is $(ED)_n$.)

<u>Theorem</u> 4. Under the same assumptions on W and D as in Theorem 1, $M=W-D$ satisfies property $(ED)_1$.

<u>Proof</u>.*) Let $\mathbb{D}_r=\{z|\ |z|<r\}$. Replacing $B[r]$ in [15] by $\mathbb{D}_r^n$, we have the following Schottky-Landau theorem.

Theorem ([15]). Assume the same assumptions on W and D as in Theorem 1. Let $f: \mathbb{D}_r^n \longrightarrow W-D$ be a holomorphic map with $J_f(0)\neq 0$. Then $r^{2n} < C\ |J_f(0)|^{-2}$, where C is a constant depending only on $f(0)$.

We proceed to the proof of Theorem 4. Assume that there exists a non-degenerate holomorphic map $f:\mathbb{C}\times\mathbb{D}^{n-1} \longrightarrow W-D$. By a translation of coordinates, we may assume that there exists a holomorphic map $f:\mathbb{C}\times\mathbb{D}_{r_0}^{n-1} \longrightarrow W-D$, with $r_0<1$, $J_f(0)\neq 0$. Define a holomorphic map $\psi: \mathbb{D}_r^n \longrightarrow \mathbb{C}\times\mathbb{D}_{r_0}^{n-1}$ by

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*) This type of argument is due to I.Nakamura.

$$\psi : (z_1,\dots,z_n) \longrightarrow ((\tfrac{r_0}{r})^{n-1}\tfrac{z_1}{a}, \tfrac{r_0}{r} z_2,\dots,\tfrac{r_0}{r} z_n),$$

where $a=|J_f(0)|$. Let $g=f\circ\psi$. Since $|J_g(0)|=|J_f(\psi(0))|\,|J_\psi(0)|=1$, we obtain a holomorphic map $g\colon \mathbb{D}_r^n \longrightarrow W-D$, with $|J_g(0)|=1$ for arbitrary $r$, which contradicts the above theorem, q.e.d.

Several degeneracy theorems are known.

<u>Theorem</u>(Fujimoto [4], Green [6]). Let $H_1,\dots,H_{n+k}$ be hyperplanes in general position in $\mathbb{P}_n$. Let $M=\mathbb{P}_n - H_1\cup\dots\cup H_{n+k}$. Then any holomorphic map $\mathbb{C}^1 \longrightarrow M$ is contained in a linear subspace of dimension $[\frac{n}{k}]$.

<u>Corollary</u>. If $k\geqq n+1$, then M satisfies property $(HD)_1$.

<u>Corllary</u>(Green [6]). Let $H_1,\dots,H_d$ be hyperplanes in $\mathbb{P}_n$ in arbitrary position. Then $M=\mathbb{P}_n - H_1\cup\dots\cup H_d$ satisfies properties $(AD)_1$ and $(ED)_n$ if $d\geqq n+2$.

<u>Corollary</u>(Green [6], p.39). Let W be a complex manifold of dimension n and let $D_1,\dots,D_k$ be hypersurfaces on W such that each $D_i\in|L|$, for a fixed line bundle L. Let $s_i$ be the section defining $D_i$. If the algebraic dimension a of $(s_1,\dots,s_k)\leqq k-2$, then $M=W-D_1\cup\dots\cup D_k$ satisfies properties $(AD)_1$ and $(ED)_{n-a+1}$.

<u>Theorem</u> (Green [6], cf. Fujimoto [5]). Let D be a Fermat variety $w_0^d+\dots+w_n^d=0$, in $\mathbb{P}_n$, where $[w_0:\dots:w_n]$ are homogeneous coordinates of $\mathbb{P}_n$. If $d>n(n+1)$, then $\mathbb{P}_n$-D satisfies properties $(AD)_1$ and $(ED)_1$ ($(ED)_1$ is a consequence of Theorem 4).

Theorem(Green [8 ]). Let D be a non-singular curve of degree d in $\mathbb{P}_2$. Let D* be the dual curve of D in the dual projective space $\mathbb{P}_2^*$. If $d \geq 3$, then $\mathbb{P}_2^* - D^*$ satisfies property $(HD)_1$.

Theorem( [15 ]). Let A be an abelian variety of dimension n and D an arbitrary hypersurface on A. Then A-D satisfies property $(ED)_n$.

Example 4. Let $X_a = \{z_1^{a_1} + \cdots + z_n^{a_n} = 0\}$ in $\mathbb{C}^n$. If $\sum_{i=1}^n \frac{1}{a_i} < 1$, then $\mathbb{C}^n - X_a$ satisfies property $(ED)_2$.

Proof. Let $U_a = \{z_1^{a_1} + \cdots + z_n^{a_n} = 1\}$. Then $\mathbb{C} \times U_a$ is an unramified covering of $\mathbb{C}^n - X_a$. By the assumption, $U_a$ satisfies property $(ED)_1$ (see [15 ]). So $\mathbb{C} \times U_a$ and $\mathbb{C}^n - X_a$ satisfy property $(ED)_2$.

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